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A Quiver and Relations for Some Group Algebras of Finite Groups

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This is a joint work with **C. Bessenrodt** and **K. Erdmann**, which is still in progress.

Here we would like to discuss on quivers with relations which come from some group algebras of finite groups over a field. Our starting point is the following purely group-theoretical theorem.

THEOREM 1. (Z^*-theorem for any prime numbers) (See [1] and [4, Theorem 4.1]). *Let p be any prime number, let G be a finite group with a Sylow p-subgroup P, and let x be any element in P. If any element $y \in P$ such that $y \neq x$ is not conjugate to x in G, then x is in the center of G modulo $O_{p'}(G)$.*

REMARK ON THEOREM 1. This is, of course, a well-known Z^*-theorem of Glauberman for $p = 2$. On the other hand, for odd primes p this can be proved only by using the classification of finite simple groups. (See [4], [1] and [2, 6.5.Theorem]).

By making use of Theorem 1 (hence, due to the classification of finite simple groups), we get the following.

THEOREM 2. (due to the classification of finite simple groups) *Let p be a prime number, and let G be a finite group such that $O_{p'}(G) = 1$, Sylow p-subgroups of G are elementary abelian, and G has a normal subgroup of index p. Then G has a suitable normal subgroup N with $G = N \times C_p$ where C_p is the cyclic group of order p.*

Now, we automaticallly obtain the next corollary on modular representation theory of finite groups by using Theorem 2. Namely,

COROLLARY 3. (due to the classification of finite simple groups) *Let K be a field of prime characteristic p, and let G be a finite group such that Sylow p-subgroups of G are elementary abelian, and G has a normal subgroup of index p. Then, G has a suitable nomal subgroup N of index p such that $B_0(KG) \cong B_0(KN) \otimes_K KC_p$ as K-algebras, where $B_0(KG)$ is the principal block ideal of the group algebra KG of G over K.*

A purpose of this note is that a similar result to Corollary 3 can be prove for the case where $p = 3$ and Sylow 3-subgroups of G are elementary abelian of order 9, say $C_3 \times C_3$, *without using the classification of finite simple groups.* In a proof there, quivers with relations (see Erdmann's book [3]), and results by Külshammer [5], [6] play an important rôle.

THEOREM 4. (Bessenrodt, Erdmann and Koshitani) **(independent from the classification of finite simple groups)** *Let K be a field of characteristic 3, and assume that G is a finite group such that Sylow 3-subgroups of G are elementary abelian $C_3 \times C_3$ of order 9, G is not 3-nilpotent, and G has a normal subgroup of index 3. Then, the principal block ideal $B_0(KG)$ of the group algebra KG is Morita equivalent to a quotient algebra $(KQ)/I$ of the path algebra KQ of a quiver Q over K with relations I, where Q has the form*

$$Q: \begin{tikzcd} \overset{e_1}{\circ} \arrow[loop left, "\gamma"] \arrow[r, shift left, "\alpha"] & \overset{e_2}{\circ} \arrow[l, shift left, "\beta"] \arrow[loop right, "\delta"] \end{tikzcd}$$

and I is an ideal of KQ generated by the relations

$$\gamma\alpha = \alpha\delta, \quad \delta\beta = \beta\gamma, \quad \alpha\beta\alpha = \beta\alpha\beta = 0, \quad \gamma^3 = \delta^3 = 0.$$

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